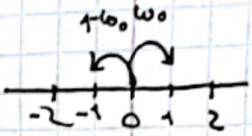
Lecture 4

Theme: One-dimensional RWRE



$\rho_x = \frac{1-\omega_x}{\omega_x}$

P – environment measure
Uniformly elliptic.
Discuss only IID case today.

$\mathbb{P}_\omega^x$ – quenched RW (starting at x in the environ. ω)

$\mathbb{P}^x$ – annealed RW (averaging over ω)

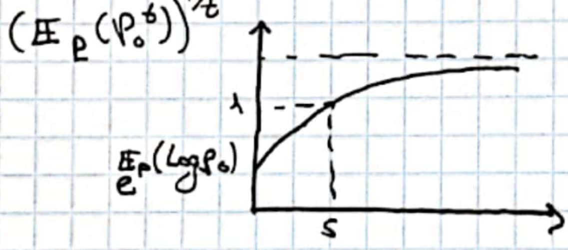
Additional facts about the transient case

Assume $\mathbb{E}_P(\log \rho_0) < 0$
(by Solomon's thm, $\mathbb{P}^0(X_n \xrightarrow{n\to\infty} +\infty) = 1$)

$T_n := \min\{k : X_k = n\}$, for $n \geq 0$

$\tau_n := T_n - T_{n-1}$ for $n \geq 1$.

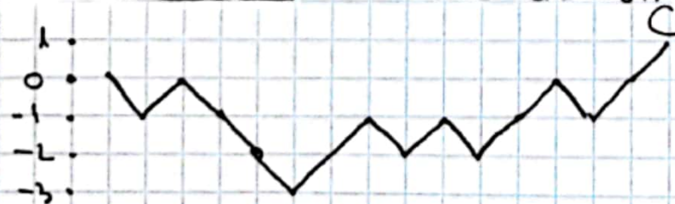
(τ_n) is a stationary and ergodic seq.



max of support of ρ_0

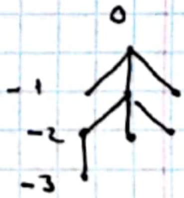
Lemma: P – IID, $\mathbb{E}_P^0(\log \rho_0) < 0$
Let $S := \sup\{t : \mathbb{E}_P(\rho_0^t) < 1\}$
Then $\mathbb{E}^0(\tau_1^t) < \infty \iff S > t$

Idea of proof: Tree encoding:



$\Longrightarrow$

$N_0 = 3$
$N_{-1} = 3$
$N_{-2} = 1$



Let N_x be the number of vertices of the tree at level $x-1$.

Then, $\tau_1 = 1 + 2\sum_{x\leq 0} N_x$. Under $\mathbb{P}_\omega^0$, each vertex of the tree at level x has a geometric$(\omega_x)-1$, number of children, indep. between vertices.
expectation $= \rho_x$

$\Longrightarrow \mathbb{E}_\omega^0(N_x) = \rho_x \, \mathbb{E}_\omega^0(N_{x+1}) = \rho_x \cdot \rho_{x+1} \cdot \ldots \cdot \rho_0.$
$N_1 := 1$

If N_x can be replaced by its expectation then

$$\tau_1 = 1 + 2(\rho_0 + \rho_0 \cdot \rho_1 + \rho_0 \rho_1 \rho_2 + \dots)$$

For each of the terms:

$$\mathbb{E}^0\left((\rho_0 \rho_1 \cdots \rho_x)^t \right) \underset{\rho \text{ IID}}{=} \left(\mathbb{E}(\rho_0^t)\right)^{|x|+1}$$

This hints why the cond. $\mathbb{E}_\rho(\rho_0^t) < 1$ is relevant to $\mathbb{E}(\tau_1^t) < \infty$.

Central limit thm.

Thm. (Annealed CLT. Kesten–Kozlov–Spitzer 1975)

Assume ρ IID, $\mathbb{E}[\log \rho_0] < 0$, $S > 2$. Then:

(a) $\frac{T_n - n v_\rho^{-1}}{\sqrt{n}}$ converges to $N(0, \sigma^2)$ for an appropriate $\sigma > 0$ (under $\mathbb{P}^0$)

(b) $\frac{X_n - n v_\rho}{\sqrt{n}}$ converges to $N(0, \sigma_a^2)$ for some $\sigma_a > 0$. (under $\mathbb{P}^0$)

(even have convergence to Brownian motion).

To prove part (a), one can use the following CLT for sums of a stationary seq.

Thm: Let $(X_n)_{n \geq 0}$ be a ~~seq~~ stationary seq. Define its strong mixing coef. by:

$$\alpha(k) := \sup_{k \geq 1} \left\{ |\operatorname{Cov}(A,B)| : \begin{array}{l} m \geq 1 \\ A \in \sigma(X_1, \dots, X_m) \\ B \in \sigma(X_{m+k}, X_{m+k+1}, \dots) \end{array} \right\}$$

Let $S_n = \sum_{j=1}^{n} X_j$. Assume, for some $\delta > 0$,

$$\mathbb{E}(|X_1|^{2+\delta}) < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \alpha(k)^{\frac{\delta}{2(2+\delta)}} < \infty$$

Then the following limit exsists:

$$\sigma^2 := \lim_{n \to \infty} \frac{\mathbb{E}[S_n^2]}{n} = \mathbb{E}[X_1^2] + 2 \sum_{j=1}^{\infty} \mathbb{E}(X_1 X_j)$$

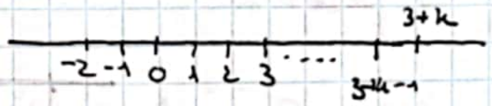
If $\sigma \neq 0$ then $\frac{S_n}{\sqrt{n}} \xrightarrow{d} N(0, \sigma^2)$

The CLT allows to prove part (a) of the prev. thm.

Write $T_n - nV_p^{-1} = \sum_{j=1}^{n} \tau_j - \mathbb{E}[\tau_j]$

If $s>2$ then $\mathbb{E}[\tau_j^{2+\delta}]<\infty$ whenever $2+\delta<s$.

It only remains to prove that $(\tau_n - \mathbb{E}[\tau_n])$ has a fast decaying $\alpha(k)$ under the annealed measure $\mathbb{P}^0$.



Example: $A \in \sigma(\tau_1, \tau_2, \tau_3)$ only influenced by environment $(\omega_n)_{n\le 2}$

$B \in \sigma(\tau_{3+k}, \tau_{3+k+1}, \ldots)$ influenced by all ω_n.

$|\mathrm{cov}(A,B)|$ is "controlled" by the event that the RW starting at ~~the~~ $3+k-1$ ever visits 2.

We calculated, in the previous lecture, for each fixed ω:

$$V_{R,L}(x) := \mathbb{P}_\omega^x(\text{Walk reaches } +R \text{ before } -L)$$

$$= \frac{\sum_{j=-L}^{x-1} \prod_{\alpha=-L+1}^{j} p_\alpha}{\sum_{j=-L}^{R-1} \prod_{\alpha=-L+1}^{p} p_y}$$

This allows to control $|\mathrm{cov}(A,B)|$ when $\mathbb{E}_P(\log p_0)<0$.

Quenched CLT

Fluctuations in $\mathbb{E}_\omega^0(X_n)$ require to replace nV_p by $\mathbb{E}_\omega^0[X_n]$. Then:

Thm. (Peterson 2008, Goldshied 2007):

P IID. $\mathbb{E}^0(\log p_0)<0$ and $s>2$.

Then for P a.e. ω, $\frac{X_n - \mathbb{E}[X_n]}{\sqrt{n}} \xrightarrow{d} N(0, \sigma_q^2)$

with $\sigma_q^2 < \sigma_a^2$.

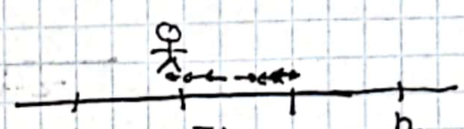
Scanned with CamScanner

Traps

How large is T_n?
Time to reach n.

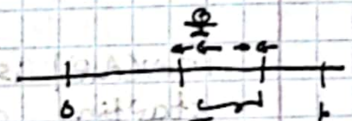
If $\mathbb{E}\tau_1 < \infty \iff (\mathbb{E}\rho_0 < 1)$ then $\frac{\mathbb{E}T_n}{n} \to v_p^{-1}$

$\iff s > 1$



T_n is dominated by strongest trap (at least when $s<1$).

What is the strength of a trap?



Recall $V_{R,-1}(0) = \mathbb{P}^0_\omega$ (reach R before -1)

$$= \frac{1}{1+\rho_0+\rho_0\rho_1+\dots+\rho_0\rho_1\cdots\rho_{R-1}}$$

heuristically, this is the number of attempts to cross a trap from 0 to R.

The walker will need $\approx 1+\rho_0+\rho_0\rho_1+\dots+\rho_0\rho_1\cdots\rho_{R-1}$ attempts to succeed.

Need to understand: $\max\limits_{0\le a\le b\le n} \underbrace{\rho_a\cdot\rho_{a+1}\cdots\rho_b}_{= e^{\sum_{j=a}^{b}\log\rho_i}}$

Cramér's thm. - Large deviations for sums of IID RVs

Let (X_n) be IID. Define $\Lambda(t) := \log\mathbb{E}(e^{tX_1})$

Assume $|\Lambda(t)| < \infty$ for all $t\in\mathbb{R}$ (this assumption can be relaxed)

Define the Legendre transform:

$$\Lambda^*(x) := \sup_{t\in\mathbb{R}}(t\cdot x - \Lambda(t))$$

Then: $\underbrace{\log\mathbb{P}\left(\sum_{j=1}^{n} X_j \ge n\cdot x\right)}/n \xrightarrow{n\to\infty} -\Lambda^*(x)$ when $x > \mathbb{E}[X_1]$

i.e $\approx \exp(-n\Lambda^*(x))$ rate function.

Easy direction: First note that when $x > \mathbb{E}(X_1)$

then $\Lambda^*(x) = \sup_{t \geq 0} (t \cdot x - \Lambda(t))$

This is due to Jensen's ineq. $\Lambda(t) \geq t\mathbb{E}[X_1]$.
and the fact that $\Lambda(0) = 0$.
For the easy direction we show

$$\mathbb{P}\left(\sum_{j=1}^{n} X_j \geq n \cdot x\right) \leq e^{-n(tx - \Lambda(t))} \quad \text{for each } t \geq 0.$$

Indeed, let $t \geq 0$. By markov's ineq.

$$\mathbb{P}\left(\sum_{j=1}^{n} X_j \geq nx\right) = \mathbb{P}\left(e^{t\sum_{j=1}^{n} X_j} \geq e^{tnx}\right) \leq$$

$$\leq \frac{\mathbb{E}[e^{t\sum_{j=1}^{n} X_j}]}{e^{tnx}} = \frac{[\mathbb{E}(e^{tX_1})]^n}{e^{tnx}} = e^{n\Lambda(t) - tnx}.$$

To understand traps, we apply the thm. to sums of $\log(\rho_i)$

The relevant expression is $\left(\sum_{j=1}^{d\log n} \log(\rho_i)\right) / \log n$ strength of a trap

We're asking for $\mathbb{P}\left(\sum_{j=1}^{d\log n} \log(\rho_i) \geq c \log n\right)$ and

seeking the biggest c s.t. this expression is roughly $1/n$

allowing to optimize over d.

(Since we have roughly n positions between 0 and n for a trap of length $\log n$) or $\frac{n}{\log n}$.

By cramer's thm. $\mathbb{P}\left(\sum_{j=1}^{d\log n} \log \rho_i \geq \underbrace{c \log n}_{\frac{c}{d} \cdot d \log n}\right) \approx e^{-d\log n \cdot \Lambda^*(c/d)}$

$= n^{-d\Lambda^*(c/d)}$.

$\Lambda(t) = \log \mathbb{E}(e^{t\log \rho_1}) = \log \mathbb{E}[\rho_1^t]$

$$\Lambda^*(t) = \sup_t (t\cdot x - \Lambda(t))$$

We seek : $\min_\alpha \alpha \Lambda^*(\frac{c}{\alpha})$

$$= \min_\alpha \alpha \cdot \sup_t (t \cdot \underset{\frac{c}{\alpha}}{\underset{\shortparallel}{x}} - \Lambda(t)) = s \cdot c$$

after calculation ($s < \infty$)

Idea of calculation:

$\sup_t$ is attained at $t(\frac{c}{\alpha})$ satisfying :

$\Lambda'(t(\frac{c}{\alpha})) = \frac{c}{\alpha}$ (by taking a derivative in t)

Now need minimum : $\min_\alpha \alpha [t(\frac{c}{\alpha}) \cdot \frac{c}{\alpha} - \Lambda(t(\frac{c}{\alpha}))] =$

$$= \min_\alpha \quad t(\tfrac{c}{\alpha}) \cdot c - \alpha \Lambda(t(\tfrac{c}{\alpha}))$$

Taking a derivative in α :

$$\underbrace{c \cdot t'(\tfrac{c}{\alpha}) \cdot (-\tfrac{c}{\alpha^2}) - \alpha \overbrace{\Lambda'(t(\tfrac{c}{\alpha}))}^{=c/\alpha} \cdot t'(\tfrac{c}{\alpha}) \cdot (-\tfrac{c}{\alpha^2})}_{=0} - \Lambda(t(\tfrac{c}{\alpha})) = 0$$

$$\Rightarrow \Lambda(t(\tfrac{c}{\alpha})) = 0. \Rightarrow t(c/\alpha) = s$$

Summary: The prob to see a trap in which

$$\sum_{j=1}^{\alpha \log n} \log p_i \geq c \cdot \log n \quad (\text{for the best } \alpha)$$

is approximately n^{-sc} $\Rightarrow$ The biggest c we will find is $\approx \frac{1}{s}$.

$\Rightarrow$ In this trap, the product of the p_i is $\approx n^{1/s}$.

When $s<1$, this dominates T_n.

i.e. $T_n \approx n^{1/s}$

When $1<s<2$ the trap takes $n^{1/s}$ time.
While T_n is of order n, the extra amount is larger than $\sqrt{n}$ and ruins the central limit thm.

Remark: When $s<2$ there is still an annealed **stable** limit theorem. (Kesten–Kozlov–Spitzer 1975)

However, there is no quenched limit thm.

The recurrent case:

$$\mathbb{E}_P(\log \rho_0) = 0 \iff \mathbb{P}^0\begin{pmatrix}\limsup X_n = \infty, \\ \liminf X_n = -\infty\end{pmatrix} = 1.$$

How big is X_n?

We exclude the homogeneous simple random walk, when $\rho_0 = 1$ deterministically.

transient case

Recurrent case is $s=0$.

It turns out that $X_n \approx (\log n)^2$.

(one heuristic is that $\left|\sum_{j=1}^{(\log n)^2} \log \rho_i\right| \approx \log n$.

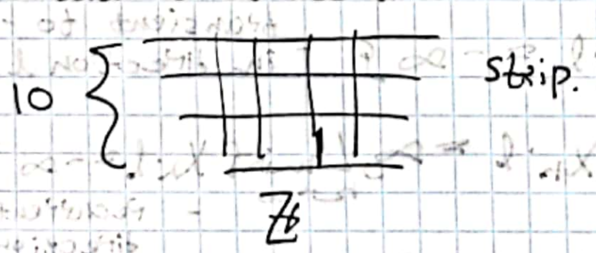
Since $\mathbb{E}\log \rho_i = 0$).

Thm. (Sinai 1982): P IID, $\mathbb{E}\log\rho_0 = 0$, ρ_0 non deterministic then $\frac{X_n}{\log^2 n}$ converges in dist. under $\mathbb{P}^0$ to

a $\underbrace{\text{non-trivial random variable}}_{\text{studied by Kesten.}}$

Extensions of one dimensional RWRE: 1) Non nearest neighbour walks on $\mathbb{Z}$

2) walks on a strip of fixed height (as in figure)



Higher-dimensional random walk in random environment

At each $x \in \mathbb{Z}^d$, have $\omega(x,e)$ for $e \in \{\pm e_i\}_{i=1}^d$, the prob. to go from x to $x+e$

P – the environment measure assumed IID and uniformly elliptic ($\omega(x,e) \geq \varepsilon > 0$ for all x, e)

$e_1 = (1,0)$

$e_2 = (0,1)$

First question: Recurrence / transience

For each environment ω (unif. elliptic), and each vector $\ell \in S^{d-1}$

Claim: $\forall x$, $\mathbb{P}^x_\omega\left(\limsup_{n\to\infty} X_n \cdot \ell \text{ finite}\right) = 0$

Indeed, when $\ell = (1,0)$ for simplicity, the event $\{\limsup_{n\to\infty} X_n \cdot \ell = k\}$ means the x coord of the walk = k inf. many times but equals $k+1$ only finitely many times. This has zero prob. by unif. ellipt. and the strong Markov prop.

Define:

$$A_\ell = \{\lim_{n\to\infty} X_n \cdot \ell = +\infty\}$$ – transient to $+\infty$ in direction ℓ.

$$A_{-\ell} = \{\lim_{n\to\infty} X_n \cdot \ell = -\infty\}$$ – transient to $-\infty$ in direction ℓ.

$$O_\ell = \{\limsup_{n\to\infty} X_n \cdot \ell = \infty, \liminf_{n\to\infty} X_n \cdot \ell = -\infty\}$$ – recurrent in direction ℓ.

By Claim, $\mathbb{P}^x_\omega(A_\ell \cup O_\ell \cup A_{-\ell}) = 1$.

$\Rightarrow \mathbb{P}^x(A_\ell \cup O_\ell \cup A_{-\ell}) = 1$.

Q: $\mathbb{P}^x(A_\ell) \in \{0,1\}$?

$\mathbb{P}^x(O_\ell) \in \{0,1\}$?

Thm (Kalikow 1981) : P IID, $\mathbb{P}^x(O_\ell) \in \{0,1\}$, Equivalently :

$$\mathbb{P}^x(A_\ell \cup A_{-\ell}) \in \{0,1\}.$$

Conjectured: P IID, $\mathbb{P}^x(A_\ell) \in \{0,1\}$

but open except when $d=2$ (Merkl - Zerner 2001).